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REGULAR SUBGRAPHS OF DENSE GRAPHS

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Every graph on n vertices, with at least $c_k n \log n$ edges contains a k-regular subgraph. This answers a question of Erdős and Sauer.

Erdős and Sauer [5] considered the following problem. Determine ex $(n, k\text{-reg}) = \max\{m$: there exists a graph on n vertices with m edges without a k-regular subgraph$\}$. They observed that ex $(n, k\text{-reg}) \leqq cn^{8/5}$ and conjectured the following: If $\varepsilon > 0$ and $n > n_0(\varepsilon, k)$ then ex $(n, k-\text{reg}) \leqq n^{1+\varepsilon}$. This has been mentioned again in the book of Bollobás [2] as an unsolved problem.

We prove the following stronger statement:

Theorem. *For every k there exists a constant c_k, such that* ex $(n, k\text{-reg}) \leqq c_k n \log n$.

Notation. $d_G(x)$ denotes the degree of the vertex $x \in V(G)$, $d(G)$ is the average degree of the graph G.

For $X \subset V(G)$, $N_G(X)$ denotes the set of all neighbours of vertices in X. For $H \subset G$ $G \setminus H$ denotes the graph obtained from G by deleting the edges of H.

Our proof is based on the following result of N. Alon, S. Friedland and G. Kalai.

Lemma 1. [1] *If q is a prime power then every graph G with maximal degree at most $2q-1$ and with at least $(q-1)n+1$ edges contains a q-regular subgraph.* ∎

Lemma 2. (Erdős [4]) *Every graph G contains a bipartite subgraph B with $|E(B)| \geqq \geqq \frac{1}{2}|E(G)|$.* ∎

Lemma 3. *Every graph G contains a bipartite subgraph H with colour classes A and B such that*

(a) H is δ-half-regular i.e. $d(x) = \delta$ for $x \in A$ and $|A| \geqq |B|$.

(b) $\delta \geqq \frac{1}{4} d(G)$.

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Proof. By Lemma 2, G contains a bipartite subgraph B_0 with $d(B_0)\geqq d(G)/2$. If there is a vertex $x\in V(B_0)$ with $d_{B_0}(x)<d(B_0)/2$ then we delete it. We then have $d(B_0\setminus x)>d(B_0)$. Repeating this procedure we obtain a graph B_1 with $d(B_1)\geqq d(B_0)$ and minimum degree $\delta\geqq d(B_1)/2$. Consider a 2-colouring (A, B) of B_1 where $|A|\geqq|B|$. All degrees of vertices in A are at least $\delta\geqq d(G)/4$ therefore $B_1\subset G$ has a subgraph H which satisfies the conditions of the Lemma. ∎

By Lemma 3 it suffices to prove that the δ-half-regular graph H contains a k regular subgraph R. R is also bipartite and as it follows from Kőnig's Theorem [3] R contains r-regular subgraphs for all $r\leqq k$. Therefore it is sufficient to deal with the case when k is a prime power.

Lemma 4. *Every δ-half-regular graph H contains a δ-half-regular subgraph with a 1-factor.*

Proof. For the colour classes A, B of H we have $|A|\geqq|B|$ by the definition. Let X be a minimal non-empty subset of A with $|N_H(X)|\leqq|X|$. Then for $v\in X$ we have $|X|-1=|X\setminus v|<|N_H(X\setminus v)|\leqq|N_H(X)|\leqq|X|$. This gives $|N_H(X)|=|X|$. By definition we have $|N_H(Y)|\geqq|Y|$ for $Y\subseteq X$. By Kőnig's Theorem there is a 1-factor in the graph induced by $X\cup N_H(X)$ in H. ∎

The proof of Theorem. Define the following series of graphs. $G_0\subset H$ is a δ-half-regular subgraph of H with a 1-factor F_0. G_i is a $(\delta-i)$-half-regular subgraph of $G_{i-1}\setminus F_{i-1}$ with a 1-factor F_i for $i\leqq\delta-1$. Indeed we have $|G|=n\geqq|G_0|\geqq|G_1|\geqq\dots$ $\dots\geqq|G_{\delta-1}|\geqq 1$.

If c_k is sufficiently large, we have

$$n<\left(\frac{2k-2}{2k-3}\right)^{\left[\frac{\delta-1}{2k-2}\right]}.$$

Consider the graphs $G_0, G_{2k-2}, \dots, G_{j(2k-2)}, \dots$. For some $j\geqq 0$ we have

$$|G_{(j+1)(2k-2)}|>\frac{2k-3}{2k-2}\,|G_{j(2k-2)}|.$$

We define the graph M as the union of the matchings $F_{j(2k-2)}, F_{j(2k-2)+1}, \dots,$ $\dots, F_{(j+1)(2k-2)}$.

The maximum degree of $M\subset G_{j(2k-2)}$ is at most $2k-1$ and

$$|E(M)|>\frac{1}{2}\,|G_{j(2k-2)}|\left(1+(2k-2)\frac{2k-3}{2k-2}\right)\geqq|M|(k-1).$$

By Lemma 1 M and consequently G contains a k-regular subgraph. ∎

Remarks 1. There is always a power of 2 between k and $2k$. Using this it is easy to see that $c_k=32k^2$ is sufficient.

Actually, using a more involved method we can prove the following theorem: For all k, there exists a constant c'_k such that if $d(G)>c'_k\log\Delta(G)$ than G contains a k-regular subgraph. On the other hand a random construction shows ex $(n$, 3-reg$)>$ $>0(n\log\log n)$. These results will be published in a forthcoming paper of Szemerédi and the author.

References

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